Research Statement

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I work in algebraic combinatorics on problems in the intersection of geometry, topology and representation theory. Combinatorics allows us to understand the 'fine structure' of fundamental objects whose existence or coarse structure is described by other means. Combinatorial problems are often synonymous with a search for new data structures, encoding the critical structure of mathematical objects in a readily-accessible form. A recurring theme in my work is the study of affine permutations, which are permutations of the integers with a kind of skew-periodicity; they arise as affine Weyl groups and have applications in juggling and many other areas. I often make use of computer exploration, and wrote the affine permutation code that appears in the Sage computer algebra system.

I choose research projects that will improve our understanding of fundamental objects that cut across many fields of mathematics. My work has already generated important new tools with ample opportunities for further results. I look forward to working with talented students to explore these results, while I press forward to establish new areas of research.

1 Canonical Decompositions of Affine Permutations and the k-Littlewood-Richardson Rule

The k-Schur functions are a generalization of Schur functions originally studied by Lapointe and Morse, arising as the cohomology of the affine Grassmannian. These functions are thus key components of the study of *affine Schubert calculus*, an affine version of a classical problem in enumerative geometry. A better understanding of the k-Schur functions would solve key problems in Macdonald theory, and allow computation of Gromov-Witten invariants and fusion coefficients. k-Schur functions are also dual to the affine Stanley symmetric functions, which are important in problems of enumerating reduced decompositioins [Lam06]. Due to the broad importance of these objects, they have received considerable attention, including an NSF Focused Research Group grant with 18 participating mathematicians [MSS$^+$12] which produced 45 research papers and forthcoming book [LLM$^+$12].

The k-Schur functions have proven extremely difficult to work with, though. The rule for multiplication of k-Schur functions is called the *k-Littlewood-Richardson rule* (k-LR rule), after an algorithm for multiplying the usual Schur functions. No complete description of the k-LR rule currently exists, even as a conjecture. Finding the general k-LR rule is a major open problem in this area.

Affine permutations are permutations of the integers satisfying a skew-periodicity rule, and can be interpreted as the Weyl group of the affine type A Lie group. The k-Schur functions can be embedded into the algebra of the affine permutation group, and the k-LR rule can then be approached as a problem on the multiplication of affine permutations. Using this approach, I developed a new canonical decomposition of affine permutations compatible with the k-Pieri rule, which is a special case of the k-LR rule[Den12]. The decomposition is into elements called *Pieri factors*, which are indexed by proper subsets of $\mathbb{Z}_{k+1}$.

Theorem 1 (Canoncial Decomposition). *Let σ be an affine permutation. Then σ has a unique maximal decomposition into Pieri factors, d_{A_i}:*

$$\sigma = d_{A_1} d_{A_2} \cdots d_{A_l}.$$

The decomposition is maximal in the sense that the vector $(|A_1|, |A_2|, \ldots, |A_m|)$ *is lexicographically maximal amongst all Pieri decompositions of* σ*. This maximal decomposition can be found directly from the* affine inversion vector *of* σ*, and can be used to directly compute the length and left descents of* σ*, and whether* σ *is fully commutative [Den13].*

This new canonical decomposition allowed the proof of a special case of the k-LR rule, which had been observed but unproven for over ten years. In particular, k-Schur functions are indexed by $k+1$-*cores*, which are partitions with no boxes of hook length $k+1$. The *boundary* of a $k+1$-core is the skew shape consisting of all boxes with hook length $< k+1$. If the boundary of a core λ is not connected, we can consider the sequence of cores $(\lambda_1, \ldots, \lambda_l)$ which when diagonally stacked give λ. We then say that λ *splits*. Then the following theorem holds:

Theorem 2 (Split Cores)**.** *Let* λ *be a* $k+1$*-core associated to a k-Schur function* $s_\lambda^{(k)}$*, and* λ *splits into* $(\lambda_1, \ldots, \lambda_l)$*. Then:*

$$s_\lambda^{(k)} = \prod s_{\lambda_i}^{(k)}$$

This theorem was proven using the realization of the k-Schur functions within the group algebra for the affine permutation group. Here, sums of certain Pieri factors emulate the generators $h_i = s_{(i)}^{(k)}$, while 'dual' Pieri factors emulate $e_i = s_{(1^i)}^{(k)}$. Using the *k-Pieri rule* [LLMS10], a well-known special case of the k-LR rule, we can find an arbitrary k-Schur function as a term appearing in a product of Pieri factors. When $i+j > k$, the product $e_i h_j$ is itself a k-Schur function; the splitting condition on cores can be used to split the k-Schur function into a parts produced from e_i's on one side and h_i's on the other. Iterating this idea gives the result. The details of the proof, however, rely heavily on the canonical decomposition of an affine permutation into Pieri factors.

Future Directions: I am developing a type-free version of the canonical decomposition, building on the type-free definition of Pieri factors by Pon [Pon12]. I expect that this study will lead to a type-free definition of the affine inversion vector, a combinatorial object which currently has no type-independent definition in spite of its usefulness in Type A.

Beyond work on the k-Schur functions, this new framework for affine permutations should have numerous applications in currently open problems. The framework seems to make Suter symmetry quite easy to understand [BZ12], and a new tableau model using the decomposition should have applications to the enumeration of reduced expressions and study of the Bruhat order and enumeration of reduced expressions.

2 Simultaneous Cores and (a, b)-Dyck Paths

A k-core partition is a partition with no *hooks* of length k. An (a, b)-core avoids hooks of both length a and b. Anderson showed that these simultaneous cores are in bijection with Dyck paths in an $a \times b$ rectangle, and thus that there are finitely many such cores [And02]. Armstrong conjectured a q-enumeration of these cores, given by the formula:

$$\frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q = \sum_t t^{\mathrm{sl}(q)+\mathrm{rank}(q)},$$

where the sum is over (a, b)-cores and the skew-length $\mathrm{sl}(t)$ is a new statistic on cores proposed by Armstrong. Armstrong, Hanusa, and Jones have proven a special case of the conjecture [AHJ13], and this and related conjectures are supported by extensive computational evidence. These objects are currently of wide interest, due to these connections with diagonal harmonics and the q,t-Catalan numbers discovered by Haglund [Hag08] and connections with the representation theory of rational

Cherednik algebras. Developing the connection between Cherednik algebras and generalized Catalan combinatorics was recently the subject of an AIM workshop [AGR$^+$12].

Future Work: In my current work, I am exploring the new skew-length statistic with Cesar Ceballos and Chris Hanusa. I have already found a simple formula for computing the skew-length from the associated Dyck path, and used this forumla to show that skew-length is symmetric in a and b, which was not obvious from the definition on cores. We have also developed the lattice path combinatorics of (a, b)-cores, and shown that skew-length is invariant under conjugation of the core, which was also non-obvious. We are also developing a new bijection on these objects which we hope will lead to a proof of Armstrong's conjecture. This bijection also provides a generalization of the reversing map on standard Dyck paths, and relates the skew-length and rank statistics with the bounce statistic developed by Haglund in his study of diagonal harmonics.

3 Monoids: Zero-Hecke, Catalan, and $\mathcal{J}$-Trivial

The *zero-Hecke monoid* can be thought of as permutations under the operation of anti-sorting: once two numbers are reverse-ordered, we no longer allow them to be put back in order. The zero-Hecke algebra $H_0(S_n)$ is obtained as the algebra of this monoid, and may also be obtained by taking the degenerate case at $q = 0$ of the Iwahori-Hecke algebra $H_q(S_n)$. The representation theory of $H_0(S_n)$ was first studied in the seventies, by Norton [Nor79] and Carter [Car86]. The zero-Hecke algebra also arises in the study of quasi-symmetric functions: Fundamental quasi-symmetric functions arise as characters of projective modules for $H_0(S_n)$ [KT97]. In my dissertation work, I described a construction for a family of orthogonal idempotents which decompose $H_0(S_n)$, providing a first solution to a problem which had been open since 1979 [Den11a].

I also demonstrated that one can also obtain a Catalan monoid as a quotient of $H_0(S_n)$, giving an algebraic structure to certain Catalan objects [Den13].

Theorem 3. *Let* Π *denote the quotient map from* $H_0(S_n)$ *to the Catalan monoid* C_n*. Then each fiber of* Π *contains a unique* [321]*-avoiding permutation of minimal length and a unique* [231]*-avoiding permutation of maximal length. This generates a bijection between* [231]*- and* [321]*-avoiding permutations, which coincides with the Simion-Schmidt bijection.*

The fibers of this quotient implement the Simion-Schmidt bijection, which is one of the fundamental early results in the study of pattern avoidance [SS85]; the quotient thus gives an algebraic framing to a previous combinatorial result. In demonstrating the quotient map, I developed a new algebraic and combinatorial interpretation of certain kinds of pattern avoidance, showing that many important cases of pattern avoidance can be interpreted through similar sorts of quotients. Finally, some of these results were explored in 'Type B' and affine analogues.

My dissertation also contained new results in the representation theory of $\mathcal{J}$-trivial monoids (a large class of monoids which includes $H_0(S_n)$) from collaboration with Hivert, Schilling and Thiéry [DHST11], and the existence of certain combinatorial crystal bases for affine type A [Den11b].

Future Work: There is ample opportunity to further study the 'width-systems' developed in my study of algebraic pattern avoidance. This may include finding the exact domain in which width-systems are useful, exploring applications to enumerative results, and finding representation-theoretic results on associated quotients.

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